STAT 3021 Lab 4

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Outline

- Newton-Leibniz formula, product rule, chain rule
- Derivative of common functions
- Densities of common continuous distributions
- Integral by substitution
- Integral by parts

Theorem 1 (Newton-Leibniz formula) *Let $F(n)$ be one of the primitive function of $f(x)$, then*

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Theorem 2 (Product rule) *Let $h(x) = f(x)g(x)$, where $f, g \in C^0$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.*

Corollary 1 (Integral by parts) $\int f dg = fg - \int g df$ *holds when either side is finite.*

Theorem 3 (Chain rule) *Let $h(x) = f(g(x))$, where $f, g \in C^0$, then $h'(x) = f'(g(x))g'(x)$.*

Corollary 2 (Integral by substitution) $\int_{[a,b]} f(g(x))g'(x)dx = \int_{[g(a),g(b)]} f(t)dt$ *(Strictly speaking, we need some conditions, though you may never need to verify them in practice.)*

Family	primitive	derivative
Power	x^a	ax^{a-1}
Log	$ln(x)$	$\frac{1}{x}$
Exponential	e^x	e^x
Trigonometric	sin(x)	cos(x)
	cos(x)	-sin(x)

Table 1: Derivative of common functions

Distribution	density
Uniform $U(a,b)$	$1_{[a,b]}$
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Exponential $\mathcal{E}(\lambda)$	$\frac{1}{\lambda} e^{-\lambda x}$

Table 2: Densities of common continuous distributions

Example 1 (Mean of standard Normal) *Calculate* $\int_0^\infty xe^{-\frac{x^2}{2}}dx$.

$$\int_0^\infty xe^{-\frac{x^2}{2}}dx = \int_0^\infty (-e^{-\frac{x^2}{2}})'dx = -e^{-\frac{x^2}{2}}|_0^\infty = 0 + 1 = 1$$

Remark: $\int_{\mathbb{R}} xe^{-\frac{x^2}{2}}dx = 0$ for it's an odd function.

Example 2 (Double integral) *Calculate* $\int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx$.

Let $S = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx$, then

$$\begin{aligned} S^2 = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy &= \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dxdy \quad \text{(Fubini Theorem)} \\ &= \int_0^{\infty} \int_0^{2\pi} re^{-\frac{r^2}{2}} d\theta dr \quad \text{(Integral by substitution)} \\ &= 2\pi \int_0^{\infty} re^{-\frac{r^2}{2}} dr = 2\pi \quad \text{(Example 1)} \end{aligned}$$

So $S = \sqrt{2\pi}$.

Example 3 (Variance of standard Normal) *Calculate* $\int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx$.

$$\begin{aligned} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx &= \int_{\mathbb{R}} -xd(e^{-\frac{x^2}{2}}) \\ &= -xe^{-\frac{x^2}{2}}|_{-\infty}^{\infty} + \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx \quad \text{(Integral by parts)} \\ &= \sqrt{2\pi} \quad \text{(Example 1)} \end{aligned}$$

Remark: So if we normalize the integral of $\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx$ by multiplying a coefficient $1/\sqrt{2\pi}$, it becomes a density function, which is exactly the density for standard normal. Besides, the result of Example 1 and 3 told us it has mean 0 and variance 1.