STAT 3021
Spring 2018
Midterm Exam (A)
Time Limit: 75 Minutes

Name (Print): SOLUTION

Student ID: ________________________

Instructions:

- Do *not* begin or turn this page until you are instructed.
- Enter all requested information on the top and bottom of this page, and put your initials on the top of every page, in case the pages become separated.
- This exam contains 7 pages (including this cover page and the multiple choice answer sheet). Check to see if any pages are missing. There are 3 multiple choice questions and 3 short answer problems.
- The exam is closed book. You may *not* use your books, or any wireless device on this exam.
- You may use a calculator and one sheet of paper (size A4 or 8.5" by 11") with formulas or other notes on both sides. You may *not* share calculators or notes!
- Show all your work on each problem for full credit except multiple choice problems. The following rules apply:
  - *Organize your work*, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
  - *Mysterious or unsupported answers will not receive full credit* for short answer problems. A correct answer, unsupported by calculations, explanation, or algebraic work will not receive full credit; an incorrect answer supported by substantially correct calculations and explanation may still receive partial credit.
  - If you need more space, use the back of the pages; clearly indicate when you have done this.

Honesty Statement and Pledge:

I have not given or received any aid or assistance to or from any other student in this course during the exam period. Everything I have written on this exam represents my own work and knowledge. I sign this knowing that infringements on the University's Academic Honest policy may result in failure or expulsion.

Signed By: ________________________ **Date:** ________________________

Problem I. Multiple Choice

Choose the ONLY ONE correct answer for each question. **Circle your answers to all questions in the answer sheet provided (page 11)**. (NO explanation is needed).

1. (2 points) For a family with two children, let A denote {first child is female}, let B denote {at least one child is female}, and let C denote {both children are female}. Select all correct statements.

   i $P(A \cap B) = \frac{1}{2}$
   ii $P(A|B) = \frac{2}{3}$
   iii $P(C|A) = P(A)$ so A and C are independent.
   iv A' and C are disjoint (mutually exclusive).

   (A) i and ii, and iii
   *** (B) **i, ii, and iv**
   (C) i, iii, and iv
   (D) ii, iii, iv

2. (2 points) Let X, Y, Z be random variables with probability distributions given below. All of them have means $\mu = 5$. Let their standard deviations be σ_X, σ_Y, and σ_Z, respectively. Sort σ_X, σ_Y, and σ_Z from the smallest to the largest.

X	3	5	6
Probability	1/4	1/4	1/2

Z	4	5	7
Probability	1/2	1/4	1/4

Y	3	7
Probability	1/2	1/2

   (A) $\sigma_X < \sigma_Y < \sigma_Z$
   (B) $\sigma_Z < \sigma_X < \sigma_Y$
   *** (C) $\sigma_X = \sigma_Z < \sigma_Y$
   (D) $\sigma_Y = \sigma_Z = \sigma_X$

3. (2 points) Which of the follow is correct statement?

   *** (A) **The unit of the standard deviation of X is the same as the unit of random variable X**
   (B) If $E(XY) = E(X)E(Y)$, then two random variables X and Y are independent.
   (C) The smaller σ^2, the greater dispersion the distribution has.
   (D) All of the above.

Problem II. Be sure to show all work for full credit.

A game costs \$10 dollars to play. The player draws 7 cards from a standard deck of cards (without replacement) and for each diamond card ($\diamondsuit$) drawn, the player gets \$4. (Note: A standard deck of playing cards consists of the four suits (diamond, club, heart, spade) and each suit contains 13 cards).

Let X be the number of diamond cards among the 7 drawn cards.

1. (4 points) What is the probability that the player will lose money?

The player loses money if he or she draws two or less diamonds, that is $P(X \leq 2) = \dfrac{\binom{13}{0}\binom{39}{7} + \binom{13}{1}\binom{39}{6} + \binom{13}{2}\binom{39}{5}}{\binom{52}{7}} = 0.7677$**.**

2. (4 points) What is the expected profit/loss from playing this game?

The expected profit/loss is $Y = 4X - 10$**. Then**

$$E(X) = \frac{(7)(13)}{52} = 1.75$$
$$E(Y) = 4E(X) - 10 = (4)(1.75) - 10 = -3$$

Therefore, the expected loss is \$3.

Problem III. Be sure to show all work for full credit.

1. (5 points) Three boys and three girls are randomly seated in a round table. What is the probability that all girls sit together and all all boys sit together.

Total # of ways to be seated: 5!=120
#of ways all boys sit together and girls sit together = (2-1)!(3!)(3!)=36.
P(all boys sit together and girls sit together) = $\frac{36}{120} = 3/10 = 0.3$

2. (5 points) Three boys and three girls are randomly seated in a round table. What is the probability that the boys and girls are seated alternatively?

Total # of ways to be seated: 5!=120
#of ways three boys sit first then girls seat between boys = (3-1)!(3!)=12.
P(boys and girls seated alternatively) = 12/120= 6/10=0.1

3. (5 points) You ask your roommate to water a sickly plant while you are on vacation. Without water, the plant will die with probability 0.8. With water, it will die with probability 0.1. The probability that your roommate will remember to water the plant is 0.85.

   What is the probability that your plant is alive when you return?

Let W **denote the event your roommate remember to water the plant.**
Let D **denote the event the plant will die**
From the description, $P(D|W) = 0.1$ **and** $P(D|W') = 0.8$
To find $P(\textbf{plant is alive}) = 1 - P(D)$

$$\begin{aligned} P(D) =& P(W \cap D) + P(W' \cap D) \\ =& P(W)P(D|W) + P(W')P(D|W') \\ =& 0.85(0.1) + 0.15(0.8) = 0.205 \end{aligned}$$

Hence

$$P(\textbf{plant will alive}) = 1 - P(D) = 1 - 0.205 = 0.795$$

OR
Let A **denote the event the plant will alive. Then**

$$\begin{aligned} P(A) &= P(W \cap A) + P(W' \cap A) \\ &= P(W)P(A|W) + P(W')P(A|W') \\ &= 0.85(0.9) + 0.15(0.2) = 0.795 \end{aligned}$$

Problem IV. Be sure to show all work for full credit.

Alice needs to catch a bus to school every morning. Let X be the waiting time for a bus in minutes after she arrives at the bus stop. The density function of X is :

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}}, & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note: $e \approx 2.71828$ is the base of the natural logarithm, which is the e on your calculator.

1. (3 points) What is the probability that Alice waits at least 10 minutes?

$$P(X \geq 10) = \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} \frac{1}{5}e^{-\frac{x}{5}}dx = (-e^{-\frac{x}{5}})|_{10}^{\infty} = e^{-2} = 0.1353$$

2. (4 points) How many minutes will she wait on average?

The expected waiting time is

$$EX = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} \frac{x}{5}e^{-\frac{x}{5}}dx = 5\int_{0}^{\infty} te^{-t}dt$$
$$= 5\int_{0}^{\infty} -tde^{-t} = 5\{(-te^{-t})|_{0}^{\infty} + \int_{0}^{\infty} e^{-t}dt\} = 5$$

Therefore, the average waiting time is 5 minutes.

3. (4 points) Given that Alice waits X minutes for the bus, it takes Y minutes for the bus to arrive at school. The conditional probability distribution is :

$$f(y|x) = \begin{cases} \frac{1}{10}, & \text{if } 10 < y < 20, x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint distribution for (X, Y). Are X and Y independent? Why? Use statistical notation/equations to support your answer.

Given the distribution of X and the conditional distribution of Y given X, we can find its joint distribution:

$$f(x, y) = \begin{cases} \frac{1}{50}e^{-\frac{x}{5}}, & \text{if } x > 0 \text{ and } 10 < y < 20 \\ 0 & \text{otherwise} \end{cases}$$

Then we could find the marginal distribution of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^{\infty} \frac{1}{50}e^{-\frac{x}{5}}dx = \frac{1}{10}, \text{ if } 10 < y < 20$$

So $f(x, y) = f(x)f_Y(y)$, then X and Y are independent.

4. (4 points) What is the expected value of Alice's total travel time in minutes (from arriving at the bus stop to school)?

The time from arriving at the bus stop to school is $Z = X + Y$. Then

$$EY = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_{10}^{20} \frac{y}{10}dy = 15$$
$$EZ = E(X + Y) = EX + EY = 5 + 15 = 20$$

So the expected time from arriving at the bus stop to school is 20 minutes.