Notation: all random variable(s) is denoted as $X_1, \cdots, X_n$.

Population distribution is X, $Z \sim N(0,1)$.

X_α is lower α-quantile of RV X.

7.11 $X \sim N(\mu, 16)$, $n=9 \Rightarrow \bar{X} \sim N(\mu, \frac{16}{9})$

$$P(|\bar{X}-\mu| \leq 2) = P\left(|Z| < \frac{2}{\sqrt{16/9}}\right)$$

$$= P\left(|Z| < \frac{3}{2}\right) = 0.866$$

7.15 $\bar{X} \sim N(\mu_1, \frac{\sigma_1^2}{m})$ $\bar{Y} \sim N(\mu_2, \frac{\sigma_2^2}{n})$

So $\bar{X}-\bar{Y} \sim N(\mu_1-\mu_2, \frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n})$ $\left(-\bar{Y} \sim N(-\mu_2, \frac{\sigma_2^2}{n})\right)$

$$P\left(\left|(\bar{X}-\bar{Y})-(\mu_1-\mu_2)\right| < 1\right) = 0.95.$$

$$\Leftrightarrow P\left(\frac{|(\bar{X}-\bar{Y})-(\mu_1-\mu_2)|}{\sqrt{\sigma_1^2/n+\sigma_2^2/n}} < \frac{1}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{n}}}\right) = 0.95$$

$$Z \stackrel{def}{=} \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\sigma_1^2/n+\sigma_2^2/n}} \sim N(0,1)$$

$$\therefore \frac{1}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{n}}} = Z_{0.025} = 1.96$$

$$n = 4.5 \times 1.96^2 \simeq 18$$

7.20 (a) $U \sim \chi^2_\nu \sim Gamma(\frac{\nu}{2}, 2) \Rightarrow \mathbb{E}U = \nu \quad Var\,U = 2\nu$

(b) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$\therefore S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$,

$\mathbb{E}S^2 = \sigma^2$, $Var\,S^2 = \frac{2\sigma^4}{n-1}$.

7.21 (a) $S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$

$P(S^2 \le b) = 0.975 \Leftrightarrow P\left(\frac{(n-1)S^2}{\sigma^2} \le \frac{b(n-1)}{\sigma^2}\right) = 0.975$

$\therefore \frac{b(n-1)}{\sigma^2} = \chi^2_{n-1, 0.975}$, $b \approx 2.42$

(b) Similarly, $\frac{a(n-1)}{\sigma^2} = \chi^2_{n-1, 0.025}$, $a \approx 0.656$

(c) 0.95.

7.26 $\frac{\bar{Y} - \mu}{S} \sim t_{n-1}$, S is sample s.d.

$\therefore P\left(\frac{g_1}{S} < t_{n-1} < \frac{g_2}{S}\right) = 0.9$

$g_1 = S\, t_{n-1; 0.05}$, $g_2 = S\, t_{n-1; 0.95}$

Notice that we assume variance is unknown.
So we introduce S and use t-distribution.

7.30 (a) $\mathbb{E}Z = 0$, $\mathbb{E}Z^2 = 1$

(b) $\mathbb{E}\dfrac{Z}{\sqrt{Y/\nu}} = \mathbb{E}Z \, \mathbb{E}\dfrac{1}{\sqrt{Y/\nu}} = 0$

$$\mathrm{Var}\frac{Z}{\sqrt{Y/\nu}} = \mathbb{E}\frac{Z^2\nu}{Y} - 0$$

$$= \mathbb{E}Z^2 \cdot \mathbb{E}\nu Y^{-1} = \frac{\nu}{\nu-2}.$$

$(\Gamma(\alpha+1) = \alpha\Gamma(\alpha))$.

7.36 (a) $S_1^2 \sim \dfrac{\sigma^2_{copper}}{9}\chi^2_9$

$S_2^2 \sim \dfrac{\sigma^2_{lead}}{7}\chi^2_7$

$\therefore \dfrac{S_1^2}{S_2^2} = 2 \; F_{9,7}$

$P\left(\dfrac{S_1^2}{S_2^2} \le b\right) = 0.95 \Leftrightarrow P\left(F_{9,7} \le \dfrac{b}{2}\right) = 0.95$

$\therefore \; b = 2F_{9,7;0.95} \approx 7.35$

(b) $a = 2F_{9,7;0.05} \approx 0.607$

(c) 0.9

7.39 (a) $\bar{X}_i \sim N(\mu_i, \frac{\sigma^2}{n_i})$

$$\therefore \hat{\theta} = \sum_{i=1}^{k} c_i \bar{X}_i \sim N\left(\sum_{i=1}^{k} c_i \mu_i, \sum_{i=1}^{k} \frac{c_i^2}{n_i}\sigma^2\right).$$

(b) $\frac{(n_i-1)S_i^2}{\sigma^2} \sim \chi^2_{n_i-1}$, S_i^2 are mutually independent,

$$\therefore \frac{SSE}{\sigma^2} = \sum_{i=1}^{k} \frac{(n_i-1)S_i^2}{\sigma^2} = \sum_{i=1}^{k} \chi^2_{n_i-1} \sim \chi^2_{n-k}.$$

(c) Let $\tau^2 = \sum_{i=1}^{k} \frac{c_i^2}{n_i}\sigma^2$, then

$$\frac{\hat{\theta}-\theta}{\sqrt{\tau^2 \frac{MSE}{\sigma^2}}} = \frac{\frac{\hat{\theta}-\theta}{\sqrt{\tau^2}}}{\sqrt{\frac{MSE}{\sigma^2}}} \sim t_{n-k}.$$

MSE doesn't follow t-distribution!

The last step is due to result in (1), (2), and the fact that $MSE \perp\!\!\!\perp \hat{\theta}$.

Independence is important.

7.43 By CLT, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, $\sigma = 2.5$, $n = 100$

$$\therefore P(|\bar{X}-\mu| < 0.5) = P\left(|Z| < \frac{0.5}{\sqrt{\sigma^2/n}}\right)$$

$$= P(|Z| < 2) \approx 0.954$$

7.44 $P\left(Z < \frac{0.4}{\sqrt{\sigma^2/n}} \right) \leq 0.95$

$\therefore \frac{0.4}{\sigma}\sqrt{n} = z_{0.95} \approx 1.96$

$n = \left(\frac{1.96}{0.4} \times 2.5 \right)^2 \approx 150$

7.58 Check that $U_n = \frac{\overline{W} - \mu_W}{\sqrt{\sigma_W^2/n}}$

7.60 $P\left(\left| \frac{\overline{X} - \overline{Y} - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \right| \leq \frac{0.05}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \right)$

$= P\left(|Z| \leq \frac{0.05}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \right) = P(|Z| \leq 2.5) \approx 0.988$

7.76 (a) $Y \sim Bin(n, p)$ $\quad Var\, Y = np(1-p)$

$Var \frac{Y}{n} = \frac{p(1-p)}{n} \stackrel{def}{=} f(p)$

Let $f'(p) = 0$, so $\frac{2p-1}{n} = 0$, $p = 0.5$.

Check that $p = 0.5$ is maximizer.

The last step is necessary for completeness. $f'(p) = 0$ itself only shows that $p = 0.5$ is a saddle point.

(b) Note that $\frac{Y}{n} = \frac{X_1 + \cdots + X_n}{n}$, $X_i \overset{i.i.d}{\sim} Ber(p)$.

So $\frac{\frac{Y}{n} - p}{\sqrt{Var \frac{Y}{n}}} \overset{CLT}{\approx} N(0,1)$

$$P\left(\left|\frac{Y}{n} - p\right| \le 0.1 \right) = 0.95$$

$$\Leftrightarrow \quad P\left(|Z| \le \frac{0.1}{\sqrt{p(1-p)/n}} \right) = 0.95$$

$$\Leftrightarrow \quad n = p(1-p)(10 Z_{0.975})^2 \approx 384\, p(1-p).$$

So n should be greater than the meximum of RHS. which is 96 when $p = 0.5$. i.e. $n \ge 96$.

Some students use $P(|\frac{Y}{n} - \mu| \le 0.1) = 0.95$, I guess they mean μ is $E\frac{Y}{n}$, which is just p.

7.84 $\frac{Y_1}{n_1} \overset{approx.}{\sim} N\left(p_1, \frac{p_1(1-p_1)}{n_1}\right)$

$\frac{Y_2}{n_2} \overset{approx.}{\sim} N\left(p_2, \frac{p_2(1-p_2)}{n_2}\right)$

Given $Y_1 \perp\!\!\!\perp Y_2$, we have

$$\frac{Y_1}{n_1} - \frac{Y_2}{n_2} \overset{approx.}{\sim} N\left(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$$

Actually. we don't need normal approximation. Can calculate exact mean and varianie.