8.1 $\mathbb{E}(\hat\theta-\theta)^2 = \mathbb{E}(\hat\theta-\mathbb{E}\hat\theta+\mathbb{E}\hat\theta-\theta)^2$

$= \mathbb{E}\left[(\hat\theta-\mathbb{E}\hat\theta)^2+2(\hat\theta-\mathbb{E}\hat\theta)(\mathbb{E}\hat\theta-\theta)+(\mathbb{E}\hat\theta-\theta)^2\right]$

$= \mathbb{E}(\hat\theta-\mathbb{E}\hat\theta)^2 + 2\underbrace{\mathbb{E}(\hat\theta-\mathbb{E}\hat\theta)}_{\to 0}\underbrace{(\mathbb{E}\hat\theta-\theta)}_{\text{constant}} + \mathbb{E}(\mathbb{E}\hat\theta-\theta)^2$

$= \operatorname{Var}\hat\theta + B^2(\hat\theta)$

8.6 a. $\mathbb{E}\hat\theta_3 = a\,\mathbb{E}\hat\theta_1+(1-a)\mathbb{E}\hat\theta_2 = a\theta+(1-a)\theta=\theta$.

b. $\because \hat\theta_1 \perp\!\!\!\perp \hat\theta_2$

$\therefore \operatorname{Var}\hat\theta_3 = a^2\operatorname{Var}\hat\theta_1+(1-a)^2\operatorname{Var}\hat\theta_2$

$= a^2\sigma_1^2+(1-a)^2\sigma_2^2 \overset{def}{=\!=} f(a)$.

Let $f'(a)=0$, we have $a\sigma_1^2-(1-a)\sigma_2^2=0$

$\therefore a = \dfrac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

8.8 $Y\sim \operatorname{Expo}(\frac{1}{\theta})$, $\mathbb{E}Y=\theta$, $\operatorname{Var}Y=\theta$.

a. $\hat\theta_1, \hat\theta_2, \hat\theta_3, \hat\theta_5$ are unbiased.

b. $\operatorname{Var}\hat\theta_1 = \operatorname{Var}Y_1=\theta$.

$\operatorname{Var}\hat\theta_2 = \frac{1}{4}(\operatorname{Var}Y_1+\operatorname{Var}Y_2) = \frac{\theta}{2}$

$\operatorname{Var}\hat\theta_3 = \frac{5}{9}\theta \qquad \operatorname{Var}\hat\theta_5=\frac{\theta}{3}$

$\therefore \hat\theta_5$ has smallest variance.

8.44 a. For $0<y<\theta$, $F_Y(y) = \int_0^y \frac{2(\theta-x)}{\theta^2}\,dx$

$= \frac{2y}{\theta}-\frac{y^2}{\theta^2}$.

b. Let $Z=\frac{Y}{\theta}$, then $f_Z(z) = \frac{\partial Y}{\partial Z} f_Y(\theta z) = 2(1-z)$, $0<z<1$.

So the distribution of Z doesn't include θ, hence is a pivotal quantity.

c. $F_Z(z) = 2z - z^2$, $0 < z < 1$

So the lower 90% quantile of Z is solved from

$$2z_\alpha - z_\alpha^2 = 0.9 \Rightarrow \quad z_\alpha = 1 - \frac{1}{\sqrt{10}}.$$

i.e. $P(Z \le z_\alpha) = 0.9$.

$\therefore$ $P(\frac{Y}{\theta} \le z_\alpha) = 0.9$, $P\left(\theta \in \left(\frac{Y}{z_\alpha}, +\infty\right)\right) = 0.9$

$\therefore$ a lower 90% confidence limit of θ is $\frac{Y}{z_\alpha}$.

8.48 $Y \sim \text{Gamma}(2, \beta)$.

a. $M_Y(t) = \mathbb{E} e^{Yt} = \int_0^\infty e^{ty} \frac{1}{\Gamma(2)\beta^2} y e^{-y/\beta} dy$

$$= (1 - \beta t)^{-2}.$$

Let $Z = \frac{2 \sum_1^n Y_i}{\beta}$

$$M_Z(t) = \mathbb{E} e^{-\frac{2}{\beta}t \sum_{i=1}^n Y_i} = \prod_{i=1}^n M_Y(\tfrac{2}{\beta}t)$$

$$= (1-2t)^{-2n},$$

which is exactly the MGF of $\text{Gamma}(2n, 2)$,

or, χ^2_{4n}.

b. $P(z_{\frac{\alpha}{2}} \le Z \le z_{1-\frac{\alpha}{2}}) = 1 - \alpha$, where $\alpha = 0.05$,

and z_α meens lower α quantile of Z.

So $P(z_{\frac{\alpha}{2}} \le \frac{2}{\beta} \sum Y_i \le z_{1-\frac{\alpha}{2}}) = 0.95$,

a 95% CI of β is $\left(\frac{2\sum Y_i}{z_{1-\frac{\alpha}{2}}}, \frac{2\sum Y_i}{z_{\frac{\alpha}{2}}}\right)$.

c. $(1.58, 5.62)$.

8.74 $P(|\bar{Y}-\mu|<0.1)=0.95$, $\sigma=0.5$

$\therefore P\left(\frac{|\bar{Y}-\mu|}{\sigma/\sqrt{n}}<\frac{0.1}{\sigma/\sqrt{n}}\right)=0.95$, $\frac{0.1}{\sigma/\sqrt{n}}=z_{0.975}$

$\therefore n\approx 96$.

It's invalid to sample from a single rainfall, since it would be dependent.

8.102 $X\sim N(\mu,\sigma^2)$.

$\hat{\mu}=57$, $\hat{\sigma}^2=144.5$, $\alpha=0.01$, $n=5$

A 99% CI of σ^2 is given by $\left(\frac{(n-1)\hat{\sigma}^2}{\chi^2_{n-1;1-\frac{\alpha}{2}}}, \frac{(n-1)\hat{\sigma}^2}{\chi^2_{n-1;\frac{\alpha}{2}}}\right)$

hence is 99% CI of $\hat{\sigma}$ is $(6.23, 52.8)$

1. $P(\hat{\theta}_1\le x)=\left(\frac{x}{\theta}\right)^4$, $f_{\hat{\theta}_1}(x)=\frac{4}{\theta}\left(\frac{x}{\theta}\right)^3$.

$\mathbb{E}\hat{\theta}_1=\int_0^\theta \frac{4}{\theta}x\cdot\left(\frac{x}{\theta}\right)^3 dx=\frac{4}{5}\theta$.

$\mathbb{E}\hat{\theta}_1^2=\int_0^\theta \frac{4}{\theta}x^2\left(\frac{x}{\theta}\right)^3 dx=\frac{2}{3}\theta^2$

$MSE(\hat{\theta}_1)=Var\,\hat{\theta}_1+(\mathbb{E}\hat{\theta}_1-\theta)^2$

$=\frac{2}{3}\theta^2-\left(\frac{4}{5}\theta\right)^2+\left(\frac{1}{5}\theta\right)^2=\frac{1}{15}\theta^2$.

$\mathbb{E}\hat{\theta}_2=\mathbb{E}2\bar{Y}=\theta$.

$Var\,\hat{\theta}_2=4\,Var\,\bar{Y}=\frac{\theta^2}{12}$.

$MSE(\hat{\theta}_2)=Var\,\hat{\theta}_2=\frac{\theta^2}{12}$

$\therefore \hat{\theta}_1$ has lower MSE.

2. $f'(x) = \sqrt{n}\,\dfrac{-x(1-x) - (q-x)\frac{1-2x}{2}}{(x(1-x))^{\frac{3}{2}}}$

$= \dfrac{\sqrt{n}}{(x(1-x))^{\frac{3}{2}}}\left(qx - \frac{x}{2} - \frac{q}{2}\right)$

Notice that $l_q(x) = qx - \frac{x}{2} - \frac{q}{2}$ is linear and

$l_q(0) = -\frac{q}{2} < 0$, $l_q(1) = \frac{q-1}{2} < 0$, for $\forall\, 0<q<1$.

So $f'(x) < 0$ for all $x \in (0,1)$, $q \in (0,1)$.

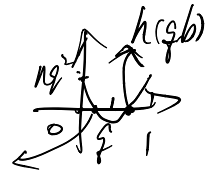
$\therefore$ $f(x)$ is strictly decreasing on $(0,1)$.

Then $-b \le f(x) \le b \iff n(q-x)^2 \le bx(1-x)$

$\iff (n+b)x^2 - (2nq+b)x + nq^2 \le 0$

, which is a quadratic function with zeros

$g(q,b) < h(q,b)$ on $(0,1)$.



3. Let $L_S = [\bar{Y} - t_{n-1;\frac{\alpha}{2}}\frac{S}{\sqrt{n}}, \bar{Y} + t_{n-1;\frac{\alpha}{2}}\frac{S}{\sqrt{n}}]$.

Because for any $S_1 < S_2$, $L_{S_1} \subset L_{S_2}$.

So $P(\mu \in L_{S_1}) \le P(\mu \in L_{S_2})$, which proves the claim.

A rigourous proof requires conditional probability.

4. Clearly, $\frac{(n_X-1)}{\sigma^2} S_X^2 \sim \chi^2_{n_X-1}$, and $S_X^2 \perp\!\!\!\perp \bar{Y} - \bar{X}$

Besides, $\bar{Y} - \bar{X} \sim N(\mu_Y - \mu_X, \frac{\sigma^2}{n_Y} + \frac{\sigma^2}{n_X})$, so $V \sim t_{n_X-1}$.

Let length of CI derived from U as L_U, we have

$$L_u = 2\, t_{n_X+n_Y-2;\frac{\alpha}{2}} \sqrt{\tfrac{1}{n_X} + \tfrac{1}{n_Y}}\, S_p \quad , \frac{n_X+n_Y-2}{\sigma^2} S_p^2 \sim \chi^2_{n_X+n_Y-2}$$

So $\mathbb{E} L_u = 2\, t_{n_X+n_Y-2;\frac{\alpha}{2}} \sqrt{1/n_X + 1/n_Y}\; \mathbb{E} S_p$

$$= 2\, t_{n_X+n_Y-2;\frac{\alpha}{2}} \sqrt{1/n_X + 1/n_Y}\; \sqrt{2}\, \frac{\Gamma\left(\frac{n_X+n_Y-1}{2}\right)}{\Gamma\left(\frac{n_X+n_Y-2}{2}\right)} \cdot \frac{\sigma}{\sqrt{n_X+n_Y-2}}$$

By same way, $L_V = 2\, t_{n_X-1;\frac{\alpha}{2}} \sqrt{1/n_X + 1/n_Y}\; S_X$

$$\mathbb{E} L_V = 2\, t_{n_X-1;\frac{\alpha}{2}} \sqrt{1/n_X + 1/n_Y}\; \sqrt{2}\, \frac{\Gamma\left(\frac{n_X}{2}\right)}{\Gamma\left(\frac{n_X-1}{2}\right)} \cdot \frac{\sigma}{\sqrt{n_X-1}}$$

When $n_X = n_Y = 10$, $\alpha = 0.05$,

$$\frac{\mathbb{E} L_u}{\mathbb{E} L_V} \approx 0.942 .$$

5. $\frac{n_X-1}{\sigma_X^2} S_X^2 \sim \chi^2_{n_X-1}$, $\frac{n_Y-1}{\sigma_Y^2} S_Y^2 \sim \chi^2_{n_Y-1}$

$\therefore \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{n_X-1, n_Y-1}$.

$\therefore P\left(F_{n_X-1, n_Y-1; \frac{\alpha}{2}} \le \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \le F_{n_X-1, n_Y-1; 1-\frac{\alpha}{2}} \right) = 1-\alpha$.

A $1-\alpha$ CI for $\frac{\sigma_X^2}{\sigma_Y^2}$ is therefore

$$\left(\frac{S_X^2/S_Y^2}{F_{n_X-1, n_Y-1; 1-\frac{\alpha}{2}}} , \frac{S_X^2/S_Y^2}{F_{n_X-1, n_Y-1; \frac{\alpha}{2}}} \right)$$