9.4. Without loss of genreality, take $\theta=1$.

$$f_{Y_{(1)}}(x)=n(1-x)^{n-1},\quad f_{Y_{(n)}}(x)=n\,x^{n-1}.$$

So
$$\begin{aligned}\operatorname{Var} Y_{(1)} &= \mathbb{E}Y_{(1)}^2-(\mathbb{E}Y_{(1)})^2\\ &=\int_0^1 n x^2(1-x)^{n-1}dx-\left(\frac{1}{n+1}\right)^2\\ &= n\cdot\frac{\Gamma(3)\Gamma(n)}{\Gamma(n+3)}-\left(\frac{1}{n+1}\right)^2\\ &=\frac{2}{(n+1)(n+2)}-\frac{1}{(n+1)^2}=\frac{n}{(n+1)^2(n+2)}\end{aligned}$$

$$\begin{aligned}\operatorname{Var} Y_{(n)} &=\int_0^1 n x^{n+1}dx-\left(\frac{n}{n+1}\right)^2=\frac{n}{n+2}-\left(\frac{n}{n+1}\right)^2\\ &=\frac{n}{(n+1)^2(n+2)}\end{aligned}$$

$\boxed{\text{So } \operatorname{Var} Y_{(1)}=\operatorname{Var} Y_{(n)}}$, hence

$$\frac{\operatorname{Var}\hat{\theta}_1}{\operatorname{Var}\hat{\theta}_2}=\frac{(n+1)^2\operatorname{Var} Y_{(1)}}{\left(\frac{n+1}{n}\right)^2\operatorname{Var} Y_{(n)}}=n^2.$$

This conclusion can be obtained by the symmetry of $Y_{(1)}$ and $Y_{(n)}$ when $Y_i \overset{i.i.d.}{\sim} \mathcal{U}([a,b])$, hence no need to compute them.

9.8. (a). $f(y)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\theta)^2}{2\sigma^2}}$, $\ln f(y)=-\frac{(y-\theta)^2}{2\sigma^2}-\ln\sqrt{2\pi}\sigma$

So $\mathbb{E}\,-\frac{\partial^2\ln f(Y)}{\partial\theta^2}=\mathbb{E}\frac{1}{\sigma^2}=\frac{1}{\sigma^2}$,

hence $I(\theta)=\left[n\,\mathbb{E}\,-\frac{\partial^2\ln f(Y)}{\partial\theta^2}\right]^{-1}=\frac{\sigma^2}{n}$.

We know that $\operatorname{Var}\bar{Y}=\frac{\sigma^2}{n}=I(\theta)$, hence $\bar{Y}$ is efficient.

(b). $p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$, $\ln p(y) = y\ln\lambda - \lambda - \ln y!$

So $\mathbb{E}\left(-\frac{\partial^2 \ln p(Y)}{\partial \lambda^2}\right) = \mathbb{E}\frac{Y}{\lambda^2} = \frac{1}{\lambda}$. $(\mathbb{E}Y = \lambda)$

$\Rightarrow I(\theta) = \frac{\lambda}{n}$.

For $Var\,\bar{Y} = \frac{1}{n} Var Y = \frac{\lambda}{n} = I(\theta)$,

we conclude $\bar{Y}$ is efficient.

9.24 (a) χ^2_n

(b) $Y_i^2 \overset{i.i.d}{\sim} \chi^2_1$, so $\mathbb{E}Y_i^2 = 1$

By WLLN, $W_n = \frac{\sum_{i=1}^n Y_i^2}{n} \xrightarrow{P} \mathbb{E}Y_i^2$

9.25 (a) $\mathbb{E}Y_1 = \mu$.

(b) $P(|Y_1 - \mu| \le 1) = P(|Z| \le 1)$, where $Z \sim N(0,1)$

≈ 0.683

(c) $P(|Y_1 - \mu| \le 1)$ is a constant doesn't go to 0

as $n \to \infty$, so Y_1 is not consistent.

9.36 $Y_n = \sum_{i=1}^n X_i$, $X_i \overset{i.i.d}{\sim}$ Bernoulli (p).

So $\frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} = \frac{Y_n - np}{\sqrt{np(1-p)}} \xrightarrow[CLT]{d} N(0,1)$ ①

Also, $\hat{p}_n = \frac{Y_n}{n} \xrightarrow[WLLN]{P} p$, hence $\hat{q}_n \xrightarrow{P} 1-p$.

Then $\hat{p}_n \hat{q}_n \xrightarrow{P} pq$. ②

Combine ①, ②, we have $\frac{\hat{p}_n - p}{\sqrt{\hat{p}_n \hat{q}_n / n}} \xrightarrow{d} N(0,1)$.

Last step due to Thm 9.3, also known as Slutsky's Thm.

9.38 (a) $$P(Y_1, \dots, Y_n \mid \mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{\sum_{i=1}^n (Y_i - \mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2 + n(\bar{Y} - \mu)^2}{2\sigma^2}} \quad (1)$$

Hence by definition, $\bar{Y}$ is sufficient for μ.

(b) Same Eq (1) tells us $\sum_{i=1}^n (Y_i - \bar{Y})^2$ is sufficient for σ^2.

(c) $$P(Y_1, \dots, Y_n \mid \mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i^2 - 2\mu Y_i + \mu^2)}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n Y_i^2 - 2\mu \sum_{i=1}^n Y_i + n\mu^2\right)}.$$

So $\sum Y_i$, $\sum Y_i^2$ are joint sufficient.

9.39. $$P(Y_1, \dots, Y_n \mid \lambda) = \prod_{i=1}^n \frac{\lambda^{y_i}}{y_i!} e^{-\lambda}$$

$$= \lambda^{\sum_{i=1}^n y_i} e^{-n\lambda} \cdot \left(\prod_{i=1}^n y_i!\right)^{-1}.$$

Hence $\sum Y_i$ is <u>SS</u>. (sufficient statistics)

9.45 Same procedure as 9.39.

9.50 $$P(Y_1, \dots, Y_n \mid \theta_1, \theta_2) = \left(\frac{1}{\theta_2 - \theta_1}\right)^n 1_{\{\theta_1 \le Y_1, \dots, Y_n \le \theta_2\}}$$

$$= \left(\frac{1}{\theta_2 - \theta_1}\right)^n 1_{\{\theta_1 \le Y_{(1)}\}} 1_{\{Y_{(n)} \le \theta_2\}}$$

So $Y_{(1)}$, $Y_{(n)}$ are SS.

9.51 Same reasoning as 9.50

9.63 (a). By results from order statistics.

(b). A SS for θ is $Y_{(n)}$,

and $$\mathbb{E} Y_{(n)} = \int_0^\theta \frac{3n y^{3n}}{\theta^{3n}} dy$$

$$= \frac{3n}{3n+1} \theta$$

Hence $\frac{3n+1}{3n} Y_{(n)}$ is MVUE.

(For it's unbiased and a function of SS).

9.65. (a) $\mathbb{E} T = P(Y_1=1, Y_2=0) = p(1-p)$

(b) $$P(T=1 \mid W=w) = \frac{P(Y_1=1, Y_2=0, W=w)}{P(W=w)}$$

$$= \frac{P(\sum_{i=3}^{n} Y_i = w-1) \cdot P(Y_1=1, Y_2=0)}{P(\sum_{i=1}^{n} Y_i = w)}$$

$$= \frac{C_{n-2}^{w-1} p^{w-1}(1-p)^{n-w-1} \cdot p(1-p)}{C_n^w p^w (1-p)^{n-w}}$$

$$= \frac{w(n-w)}{n(n-1)}$$

(c). $$\mathbb{E}(T \mid W) = P(T=1 \mid W) = \frac{W(n-W)}{n(n-1)} = \frac{n}{n-1} \cdot \frac{W}{n} \cdot (1-\frac{W}{n})$$

$$= \frac{n}{n-1} \bar{Y}(1-\bar{Y})$$

9.72. $\hat{\mu} = \bar{Y}$ (sample mean)

$\widehat{\mathbb{E}Y^2} = \frac{\sum Y_i^2}{n}$ (sample 2-nd moment)

So $\hat{\sigma}^2 = \widehat{\mathbb{E}Y^2} - \hat{\mu}^2 = \frac{\sum Y_i^2}{n} - (\bar{Y})^2$.

9.80 (a) $P(Y_1, \dots, Y_n | \lambda) = \frac{\lambda^{\sum Y_i}}{\prod Y_i!} e^{-n\lambda}$.

So log-likelihood is

$l_n(\lambda) = \log P(Y_1 \dots, Y_n|\lambda) = \sum Y_i \ln\lambda - n\lambda + \ln \prod Y_i!$

Let $l_n'(\lambda) = 0$, we have $\hat{\lambda} = \frac{\sum Y_i}{n}$.

(b) $\mathbb{E}\hat{\lambda} = \mathbb{E}Y = \lambda$, $Var\hat{\lambda} = \frac{Var Y}{n} = \frac{\lambda}{n}$.

(c) By WLLN, $\hat{\lambda} \xrightarrow{P} \lambda$.

(d). For the mapping $\lambda \to e^{-\lambda}$ is monotone,

so MLE of $e^{-\lambda}$ is $e^{-\hat{\lambda}}$.

9.85 (a) $f(Y_1, \dots, Y_n | \alpha, \theta) = \left(\frac{1}{\Gamma(\alpha)\theta^\alpha}\right)^n e^{-\frac{\sum y_i}{\theta}} \prod_{i=1}^n y_i^{\alpha-1}$

$l_n(\theta) = -n\alpha \ln\theta - \frac{\sum y_i}{\theta} + c$,

c is a constant not involve unknow parameter θ.

So let $l_n'(\theta) = 0$, we have $\hat{\theta} = \frac{\sum y_i}{n\alpha}$.

(b) $\mathbb{E}Y = \alpha\theta$, $Var Y = \alpha\theta^2$, so

$\mathbb{E}\hat{\theta} = \theta$, $Var\hat{\theta} = \frac{\theta^2}{n\alpha}$.

(c) By WLLN, $\hat{\theta} \xrightarrow{P} \theta$.

Alternatively, for $Var\hat{\theta}_n = \frac{\theta}{n} \to 0$, by Markov inequality we have that for any $\varepsilon > 0$,

$$P(|\hat{\theta}_n - \theta| > \varepsilon) \le \frac{Var\hat{\theta}_n}{\varepsilon^2} = \frac{\theta}{n\varepsilon^2} \to 0.$$

9.86. $P(X_1, \cdots, X_m, Y_1, \cdots, Y_n | \mu_1, \mu_2, \sigma^2)$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{m+n} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(X_i-\mu_1)^2 + \sum_{i=1}^{n}(Y_i-\mu_2)^2\right]}$$

So $\ln(\sigma^2, \mu_1, \mu_2) = -\frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(X_i-\mu_1)^2 + \sum_{i=1}^{n}(Y_i-\mu_2)^2\right] - \frac{m+n}{2}\ln\sigma^2$.

$$\frac{\partial \ln(\sigma^2, \mu_1, \mu_2)}{\partial \mu_1} = 0 \Rightarrow \hat{\mu}_1 = \bar{X} = \frac{\sum X_i}{m}$$

Similarly. $\frac{\partial \ln(\sigma^2, \mu_1, \mu_2)}{\partial \mu_2} = 0 \Rightarrow \hat{\mu}_2 = \bar{Y}$.

Therefore, $\hat{\sigma}^2$ satisfies $\frac{\partial \ln(\sigma^2, \hat{\mu}_1, \hat{\mu}_2)}{\partial(\sigma^2)} = 0$,

which yields $\hat{\sigma}^2 = \frac{1}{m+n}\left[\sum_{i=1}^{m}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(Y_i-\bar{Y})^2\right]$.

9.94 Let likelihood function be $l(\theta)$,

then $l(\hat{\theta}) = \sup_{\theta\in\Theta} l(\theta)$.

So likelihood function for $t(\theta)$ is $l(t^{-1}(t(\theta)))$

and $l(t^{-1}(t(\hat{\theta}))) = l(\hat{\theta}) = \sup_{\theta\in\Theta} l(\theta) \ge l(t^{-1}(t(\theta)))$

hence $t(\hat{\theta})$ is MLE of $t(\theta)$.

1. Solution 1:

For $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} y$, so for any $\varepsilon > 0$, $\exists N > 0$,

s.t. $\forall n > N$, we have $P(|X_n - x| > \varepsilon) \leq \varepsilon$, $P(|Y_n - y| > \varepsilon) \leq \varepsilon$.

So $\forall n > N$,
$$
\begin{aligned}
|X_n Y_n - xy| &\leq |X_n Y_n - X Y_n| + |x Y_n - xy| \\
&\leq |X_n - X|\,|Y_n| + |X|\,|Y_n - y| \\
&\leq \varepsilon(|y| + \varepsilon) + \varepsilon |X| \\
&\leq \varepsilon M. \quad (M \text{ is a constant}).
\end{aligned}
$$

holds with prob at least $1 - 2\varepsilon$.

Therefore, $X_n Y_n \xrightarrow{P} xy$.

Solution 2:

Verify that if $X \xrightarrow{P} X$, then $X^2 \xrightarrow{P} X^2$.

So $(X_n + Y_n)^2 \xrightarrow{P} (x+y)^2$, $(X_n - Y_n)^2 \xrightarrow{P} (x-y)^2$,

hence $X_n Y_n = \dfrac{(X_n + Y_n)^2 - (X_n - Y_n)^2}{4} \xrightarrow{P} \dfrac{(x+y)^2 - (x-y)^2}{4} = xy$.