10.75 The tremendous growth of the Florida lobster (called *spiny lobster*) industry over the past 20 years has made it the state's second most valuable fishery industry. A declaration by the Bahamian government that prohibits U.S. lobsterers from fishing on the Bahamian portion of the continental shelf was expected to reduce dramatically the landings in pounds per lobster trap. According to the records, the prior mean landings per trap was 30.31 pounds. A random sampling of 20 lobster traps since the Bahamian fishing restriction went into effect gave the following results (in pounds):

17.4	18.9	39.6	34.4	19.6
33.7	37.2	43.4	41.7	27.5
24.1	39.6	12.2	25.5	22.1
29.3	21.1	23.8	43.2	24.4

Do these landings provide sufficient evidence to support the contention that the mean landings per trap has decreased since imposition of the Bahamian restrictions? Test using $\alpha = .05$.

Sol: $n=20$, $\bar{X}=28.94$, sample sd $s=9.50$,

$H_0: \mu \geq 30.31$ $\quad H_1: \mu < 30.31 \stackrel{\text{def}}{=} \mu_0$

Assume $X_i \stackrel{i.i.d.}{\sim} N(\mu, \sigma^2)$. $\quad RR: \{T < t_{n-1;\alpha}\}$.

Test statistic $T = \dfrac{\bar{X}-\mu}{s/\sqrt{n}} = -0.647$

When $\alpha = 0.05$, $t_{n-1;\alpha} = -1.729$.

$\because T > t_{n-1;\alpha}$

$\therefore$ We cannot reject null hypothesis.

—

X

10.93 For a normal distribution with mean μ and variance $\sigma^2 = 25$, an experimenter wishes to test $H_0 : \mu = 10$ versus $H_a : \mu = 5$. Find the sample size n for which the most powerful test will have $\alpha = \beta = .025$.

$\bar{X}$ $\quad \varphi(\bar{X}) = 1$ choose H_a

LRT

10.93 The UMP test is

$$\varphi(\bar{X}) = \begin{cases} 1 & \bar{X} < c', \\ 0 & \bar{X} > c', \end{cases}$$

where c' is determined by $P_{H_0}(\bar{X} < c') = \alpha$,

i.e. $P\left(\frac{\bar{X}-10}{\sigma/\sqrt{n}} < c\right) = \alpha$, $\quad c = -Z_\alpha$

Under H_1, $\beta = P_{H_1}\left(\frac{\bar{X}-10}{\sigma/\sqrt{n}} > c\right) = P_{H_1}\left(\frac{\bar{X}-5}{\sigma/\sqrt{n}} > c + \frac{5}{\sigma/\sqrt{n}}\right)$

$$= P(Z > c + \sqrt{n})$$

Given $\alpha = \beta = 0.025$, we have $\quad n = 4 Z_\alpha^2$

$$= 15.4$$

$\therefore$ $n = 16$

10.94 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with *known* mean μ and unknown variance σ^2. Find the most powerful α-level test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_0^2$. Show that this test is equivalent to a χ^2 test. Is the test uniformly most powerful for $H_a : \sigma^2 > \sigma_0^2$?

10.94 The LRT is

$$\varphi(S_Y) = \begin{cases} 1 & S_Y^2 \geq c \\ 0 & S_Y^2 < c \end{cases}, \quad \left(S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right)$$

where c is determined by

$P_{H_0}(S_Y^2 \geq c) = \alpha$. Since $S_Y^2 \sim \sigma^2 \chi^2_{n-1}$,

So under H_0, $P\left(\frac{S_Y^2}{\sigma_0^2} \geq c\right) = \alpha$, $c = \chi^2_{n-1;\alpha}$.

It's most powerful for any $\sigma_1 > \sigma_0$, hence it's UMP for $H_1 : \sigma^2 > \sigma_0^2$.

Since μ is known, if we use $S_Y^2 = \frac{1}{n}\sum(Y_i - \mu)^2$, then $S_Y^2 \sim \chi^2_n$, and c should be $\chi^2_{n,\alpha}$.

Apologize if I take off 1 pts for my carelessness.

10.108 Suppose that $X_1, X_2, \ldots, X_{n_1}, Y_1, Y_2, \ldots, Y_{n_2}$, and $W_1, W_2, \ldots, W_{n_3}$ are independent random samples from normal distributions with respective unknown means μ_1, μ_2, and μ_3 and variances σ_1^2, σ_2^2, and σ_3^2.

a Find the likelihood ratio test for $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2$ against the alternative of at least one inequality.

b Find an approximate critical region for the test in part (a) if n_1, n_2, and n_3 are large and $\alpha = .05$.

10.108 Under H_0, the joint density / likelihood is

$$L(\mu_1, \mu_2, \mu_3, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n_1+n_2+n_3} e^{-\frac{1}{2\sigma^2}\left(\Sigma(X_i-\mu_1)^2 + \Sigma(Y_i-\mu_2)^2 + \Sigma(W_i-\mu_3)^2\right)}$$

The MLE is given by $\hat{\mu}_1 = \bar{X}$, $\hat{\mu}_2 = \bar{Y}$, $\hat{\mu}_3 = \bar{W}$,

$$\hat{\sigma}^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2 + (n_3-1)S_Z^2}{n_1+n_2+n_3}.$$

So $L_{H_0} = (2\pi\hat{\sigma}^2)^{-\frac{n_1+n_2+n_3}{2}}$.

Under H_1, the likelihood

$$L(\mu_1, \mu_2, \mu_3, \sigma_1^2, \sigma_2^2, \sigma_3^2) = \left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^{n_1} e^{-\frac{1}{2\sigma_1^2}\Sigma(X_i-\mu_1)^2}$$
$$\times \left(\frac{1}{\sqrt{2\pi}\sigma_2}\right)^{n_2} e^{-\frac{1}{2\sigma_2^2}\Sigma(Y_i-\mu_2)^2}$$
$$\times \left(\frac{1}{\sqrt{2\pi}\sigma_3}\right)^{n_3} e^{-\frac{1}{2\sigma_3^2}\Sigma(W_i-\mu_3)^2}$$

$\tilde{\mu}_1 = \bar{X}$, $\hat{\mu}_2 = \bar{Y}$, $\tilde{\mu}_3 = \bar{W}$, $\tilde{\sigma}_1^2 = \frac{n_1-1}{n_1}S_X^2$, $\tilde{\sigma}_2^2 = \frac{n_2-1}{n_2}S_Y^2$,

$\tilde{\sigma}_3^2 = \frac{n_3-1}{n_3}S_W^2$.

$$L_{H_1} = (2\pi\tilde{\sigma}_1^2)^{-\frac{n_1}{2}}(2\pi\tilde{\sigma}_2^2)^{-\frac{n_2}{2}}(2\pi\tilde{\sigma}_3^2)^{-\frac{n_3}{2}}$$

$$LR = \frac{\hat{\sigma}_1^{n_1}\hat{\sigma}_2^{n_2}\hat{\sigma}_3^{n_3}}{\hat{\sigma}^{n_1+n_2+n_3}} \qquad LR = \frac{L_{H_0}}{L_{H_1}}$$

LRT is given by

$$\varphi(X,Y,W) = \begin{cases} 1 & LR \le c \\ 0 & LR > c \end{cases},$$

$$P_{H_0}(LR \le c) = \alpha. \rightarrow P_{H_0}(\underbrace{-2\ln(LR)}_{}\ge -2\ln c) = \alpha$$

(b) By Thm 10.2, $-2\ln(LR) \underset{approx}{\overset{H_0}{\sim}} \chi^2_2$ $\quad -2\ln c = \chi^2_{2;\alpha}$

$$\therefore\ c = e^{-\frac{1}{2}\chi^2_{2;\alpha}}$$

Remark: Thm 10.2 NOT always holds.

Just like MLE is NOT always consistent.

Example: $X_{ij} \sim N(\mu_i, \sigma^2)$, $i=1,2,\cdots,n$, $j=1,2$,

then MLE of σ^2 is $\hat{\sigma}^2 = \frac{1}{n}\cdot\frac{1}{2}\cdot\sum_i\sum_j (X_{ij}-\bar{X}_i)^2$

So $\hat{\sigma}^2 \sim \frac{\sigma^2}{2}\frac{\chi^2_n}{n} \xrightarrow[p]{WLLN} \frac{\sigma^2}{2}$.

10.111 Suppose that we are interested in testing the *simple* null hypothesis $H_0 : \theta = \theta_0$ versus the *simple* alternative hypothesis $H_a : \theta = \theta_a$. According to the Neyman–Pearson lemma, the test that maximizes the power at θ_a has a rejection region determined by

$$\frac{L(\theta_0)}{L(\theta_a)} < k.$$

In the context of a likelihood ratio test, if we are interested in the *simple* H_0 and H_a, as stated, then $\Omega_0 = \{\theta_0\}$, $\Omega_a = \{\theta_a\}$, and $\Omega = \{\theta_0, \theta_a\}$.

a Show that the likelihood ratio λ is given by

$$\lambda = \frac{L(\theta_0)}{\max\{L(\theta_0), L(\theta_a)\}} = \frac{1}{\max\left\{1, \dfrac{L(\theta_a)}{L(\theta_0)}\right\}}.$$

b Argue that $\lambda < k$ if and only if, for some constant k',

$$\frac{L(\theta_0)}{L(\theta_a)} < k'.$$

c What do the results in parts (a) and (b) imply about likelihood ratio tests when both the null and alternative hypotheses are simple?

10.111 a. By definition.

b. $\lambda < k \iff \dfrac{1}{\max\{1, \frac{L(\theta_a)}{L(\theta_0)}\}} < k$ (note $0 < k \leq 1$)

$\iff \max\{1, \frac{L(\theta_a)}{L(\theta_0)}\} > \frac{1}{k}$

$\iff \dfrac{L(\theta_a)}{L(\theta_0)} > \dfrac{1}{k}$

$\iff \dfrac{L(\theta_0)}{L(\theta_a)} < k$.

c. It implies LRT is UMP in simple case.