Nov. 17th.

Hypothesis testing.

1. Point estimation Revisit.
2. Connection with hypothesis testing
3. Section 10.8, 10.9 (Small sample)

Setting: Population distribution $X \sim F_X$. <u>unknown</u>

Samples $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} F_X$. $\mu = \mathbb{E}X$, $\sigma^2 = \operatorname{Var} X$

Estimation. $\hat{\mu} = \bar{X} = \dfrac{\sum X_i}{n}$, $\hat{\sigma}^2 = \dfrac{1}{\ \ 1}\sum (X_i - \bar{X})^2$.

Try to figure out the distribution of the estimator, which is the core of estimation theory.

① F_X is normal $N(\mu, \sigma^2)$.

Normal distribution has nice properties.

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

- $\mathbb{E}\bar{\mu}$, $\operatorname{Var}\bar{\mu}$
- $P(\underbrace{|\bar{\mu} - \mu| < a}) = p$.
- CI

$P(\mu \in [\ ,\ ]) = 1-\alpha$.

$\Uparrow$

$P(\underbrace{\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \in [-t_{n-1,\frac{\alpha}{2}}, t_{n-1,\frac{\alpha}{2}}]}) = 1-\alpha$

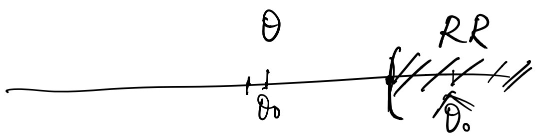
$\dfrac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$, $\bar{X} \perp\!\!\!\perp \hat{\sigma}^2$.

$$\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} = \frac{(\bar{X}-\mu)/\sigma/\sqrt{n}}{\hat{\sigma}/\sigma} \sim t_{n-1}$$

Sufficiency.

Hypothesis testing:

$H_0: \theta = \theta_0$ v.s. $H_1: \begin{cases} \theta > \theta_0 \\ \theta < \theta_0 \\ \theta \neq \theta_0 \end{cases}$



$\theta = EX = \mu$

$\bar{X} = \hat{\mu}$

Rejection Region & test statistics T

$P(T \in RR \mid H_0 \text{ is true}) \leq \alpha \to$ type I error / significance level.

$P\left(\frac{\hat{\theta} - \theta}{S/\sqrt{n}} \in \underbrace{[t_{1-\frac{\alpha}{2}; n-1}, +\infty)}_{RR} \right) = \alpha.$

A Small-Sample Test for μ

Assumptions: $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with $E(Y_i) = \mu$.

$H_0: \mu = \mu_0$.

$$H_a: \begin{cases} \mu > \mu_0 & \text{(upper-tail alternative).} \\ \mu < \mu_0 & \text{(lower-tail alternative).} \\ \mu \neq \mu_0 & \text{(two-tailed alternative).} \end{cases}$$

Test statistic: $T = \dfrac{\overline{Y} - \mu_0}{S/\sqrt{n}}$.

$$\text{Rejection region: } \begin{cases} t > t_\alpha & \text{(upper-tail RR).} \\ t < -t_\alpha & \text{(lower-tail RR).} \\ |t| > t_{\alpha/2} & \text{(two-tailed RR).} \end{cases}$$

Small-Sample Tests for Comparing Two Population Means

Assumptions: Independent samples from normal distributions with $\sigma_1^2 = \sigma_2^2$.

$H_0: \mu_1 - \mu_2 = D_0$.

$$H_a: \begin{cases} \mu_1 - \mu_2 > D_0 & \text{(upper-tail alternative).} \\ \mu_1 - \mu_2 < D_0 & \text{(lower-tail alternative).} \\ \mu_1 - \mu_2 \neq D_0 & \text{(two-tailed alternative).} \end{cases}$$

Test statistic: $T = \dfrac{\overline{Y}_1 - \overline{Y}_2 - D_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$, where $S_p = \sqrt{\dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$.

$$\text{Rejection region: } \begin{cases} t > t_\alpha & \text{(upper-tail RR).} \\ t < -t_\alpha & \text{(lower-tail RR).} \\ |t| > t_{\alpha/2} & \text{(two-tailed RR).} \end{cases}$$