Sept. 28th.

Outline:

① Point estimation, order statistics, Gamma dist

② Confidence interval.

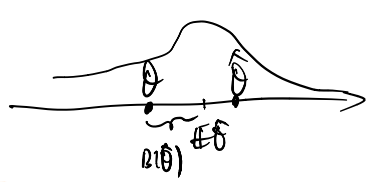
8.1 Using the identity

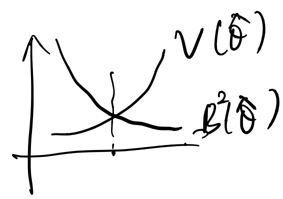
$$(\hat{\theta} - \theta) = [\hat{\theta} - E(\hat{\theta})] + [E(\hat{\theta}) - \theta] = [\hat{\theta} - E(\hat{\theta})] + B(\hat{\theta}),$$

show that

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + (B(\hat{\theta}))^2.$$

$$\begin{aligned}
\mathbb{E}(\hat{\theta}-\theta)^2 &= \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta} + \mathbb{E}\hat{\theta} - \theta)^2 \\
&= \mathbb{E}\left[(\hat{\theta} - \mathbb{E}\hat{\theta})^2 + 2(\hat{\theta} - \mathbb{E}\hat{\theta})(\mathbb{E}\hat{\theta} - \theta) + (\mathbb{E}\hat{\theta} - \theta)^2\right] \\
&= \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2 + 2\,\mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})(\mathbb{E}\hat{\theta} - \theta) + \mathbb{E}(\mathbb{E}\hat{\theta} - \theta)^2 \\
&= \text{Var}\,\hat{\theta} + B^2(\hat{\theta})
\end{aligned}$$





8.8 Suppose that Y_1, Y_2, Y_3 denote a random sample from an exponential distribution with density function

$$f(y) = \begin{cases} \left(\dfrac{1}{\theta}\right) e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the following five estimators of θ:

$$\hat{\theta}_1 = Y_1, \qquad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \qquad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \qquad \hat{\theta}_4 = \min(Y_1, Y_2, Y_3), \qquad \hat{\theta}_5 = \overline{Y}.$$

a Which of these estimators are unbiased?

b Among the unbiased estimators, which has the smallest variance?

$\mathrm{Expo}(\theta)$ has pdf $f(x) = \theta e^{-\theta x}$, $x \geq 0$.

$Y_1, Y_2, Y_3 \sim \mathrm{Expo}(\frac{1}{\theta})$. If $X \sim \mathrm{Expo}(\theta)$, $\mathbb{E}X = \frac{1}{\theta}$.

$\mathrm{Var}\,X = \frac{1}{\theta^2}$.

$\mathbb{E}Y_1 = \mathbb{E}Y_2 = \mathbb{E}Y_3 = \theta$.

$\mathrm{Var}\,Y_1 = \theta^2$.

a. $\hat{\theta}_1 = Y_1$, $\mathbb{E}Y_1 = \theta$.

$\hat{\theta}_2 = \frac{Y_1+Y_2}{2}$, $\mathbb{E}\frac{Y_1+Y_2}{2} = \frac{1}{2}\mathbb{E}(Y_1+Y_2) = \frac{1}{2}(\mathbb{E}Y_1 + \mathbb{E}Y_2) = \theta$

$\hat{\theta}_3 = \frac{Y_1+2Y_3}{3}$

$\hat{\theta}_4 = \min(Y_1, Y_2, Y_3) = Y_{(1)}$

$\hat{\theta}_5 = \bar{Y}$

$X_1, \dots, X_n \overset{i.i.d.}{\sim} X$. $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$

pdf of X is $f(x)$, cdf is $F(x)$ $f(x) = F'(x)$

$P(X_{(1)} \geq x) = P(X_1, \dots, X_n \geq x) = \prod_{i=1}^{n} P(X_i \geq x)$

$= 1 - F_{X_{(1)}}(x)$ $= (1 - F(x))^n$

$P(X_{(n)} \leq x) = P(X_1, \dots, X_n \leq x) = F^n(x)$.

$\hat{\theta}_4 = Y_{(1)}$ $F_Y(y) = e^{-\frac{y}{\theta}}$

$\hat{\theta}_4 = \min(Y_1, Y_2, Y_3) \geq Y_1$

$F_{\hat{\theta}_4}(y) = 1 - (1 - F_Y(y))^3$

$\mathbb{E}\hat{\theta}_4 > \mathbb{E}Y_1 = \theta$.

$f_{\hat{\theta}_4}(y) = 3 f_Y(y)(1 - F_Y(y))^2$.

$= 3\frac{1}{\theta} e^{-\frac{y}{\theta}} (1 - e^{-\frac{y}{\theta}})^2$.

***8.16** Suppose that $Y_1, Y_2, \dots, Y_n$ constitute a random sample from a normal distribution with parameters μ and σ^2.[1]

a Show that $S = \sqrt{S^2}$ is a biased estimator of σ. [*Hint:* Recall the distribution of $(n-1)S^2/\sigma^2$ and the result given in Exercise 4.112.]

b Adjust S to form an unbiased estimator of σ.

c Find an unbiased estimator of $\mu - z_\alpha\sigma$, the point that cuts off a lower-tail area of α under this normal curve.

$X \sim \text{Gamma}(\alpha, \beta)$ pdf $f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}$. $\int_0^{\infty} f(x)dx = 1$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

$\mathbb{E}X = \alpha\beta$, $\text{Var}\, X = \alpha\beta^2$.

$\text{Expo}(\theta) \sim \text{Gamma}(1, \frac{1}{\theta})$ $\qquad Y \sim N(\mu, \sigma^2)$

$\chi^2_n \sim \text{Gamma}(\frac{n}{2}, 2)$

a. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \sim \text{Gamma}(\frac{n-1}{2}, 2) \sim X$

$$\mathbb{E}\, S = \mathbb{E}\,(S^2)^{\frac{1}{2}} = \mathbb{E}\left(\frac{(n-1)S^2}{\sigma^2}\right)^{\frac{1}{2}} \cdot \left(\frac{n-1}{\sigma^2}\right)^{-\frac{1}{2}}$$

$$= \frac{\sigma}{\sqrt{n-1}} \mathbb{E}\, X^{\frac{1}{2}}$$

$$= \frac{\sigma}{\sqrt{n-1}} \int_0^{\infty} x^{\frac{1}{2}} \cdot \frac{1}{\Gamma(\frac{n-1}{2}) 2^{\frac{n-1}{2}}} x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \frac{\sigma}{\sqrt{n-1}} \cdot \frac{\Gamma(\frac{n-1}{2}+\frac{1}{2}) 2^{\frac{n-1}{2}+\frac{1}{2}}}{\Gamma(\frac{n-1}{2}) 2^{\frac{n-1}{2}}}$$

$$= \boxed{\frac{\Gamma(\frac{n-1}{2}+\frac{1}{2})}{\Gamma(\frac{n-1}{2})} \cdot \sqrt{\frac{2}{n-1}}}\ \sigma. \longrightarrow C_n.$$

b. $\mathbb{E}\, \frac{1}{C_n} S = \sigma$.

c. $\hat{\mu} - Z_\alpha \hat{\sigma}$, $\quad \bar{Y} - Z_\alpha \frac{S}{C_n}$

8.43 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population with a uniform distribution on the interval $(0, \theta)$. Let $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ and $U = (1/\theta)Y_{(n)}$.

a Show that U has distribution function

$$F_U(u) = \begin{cases} 0, & u < 0, \\ u^n, & 0 \le u \le 1, \\ 1, & u > 1. \end{cases}$$

b Because the distribution of U does not depend on θ, U is a pivotal quantity. Find a 95% lower confidence bound for θ.

a. $F_Y(y) = \frac{y}{\theta}$, $f_Y(y) = \frac{1}{\theta} 1_{\{y \le \theta\}}$.

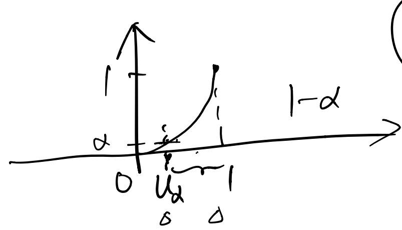
$F_{Y_{(n)}}(y) = F_Y^n(y) = (\frac{y}{\theta})^n$, $0 \le y \le \theta$

$U = \frac{Y_{(n)}}{\theta}$, $F(U \le u) = F(\frac{Y_{(n)}}{\theta} \le u)$

$= F_{Y_{(n)}}(u\theta) = u^n$, $0 \le u \le 1$

b. $P(\underbrace{\frac{Y_{(n)}}{\theta}}_{U} \in \boxed{[u_\alpha, \infty)}) = 0.95$

$= P(\theta \in (0, \frac{Y_{(n)}}{u_\alpha}]) = 0.95$



$U = \frac{Y_{(n)}}{\theta}$

$P(U \in [u_\alpha, \infty)) = 0.95$

$= P(\theta \in (0, \frac{Y_{(n)}}{u_\alpha}]) = 0.95$

8.46 Refer to Example 8.4 and suppose that Y is a single observation from an exponential distribution with mean θ. $Y \sim Expo(\frac{1}{\theta})$

a Use the method of moment-generating functions to show that $2Y/\theta$ is a pivotal quantity and has a χ^2 distribution with 2 df.

b Use the pivotal quantity $2Y/\theta$ to derive a 90% confidence interval for θ.

a. $M_Y(t) = \mathbb{E} e^{Yt} = \int_0^\infty e^{yt} \cdot \frac{1}{\theta} e^{-\frac{y}{\theta}} dy$.

$= (t - \frac{1}{\theta})^{-1} \cdot \frac{1}{\theta} = \frac{1}{1-\theta t}$.

$= (1-\theta t)^{-1}$

$Gamma(1, \theta)$.

$\chi^2_n \sim Gamma(\frac{n}{2}, 2)$

$Z = \frac{2Y}{\theta}$

$M_Z(t) = \mathbb{E} e^{\frac{2Y}{\theta} t} = \mathbb{E} e^{\frac{2t}{\theta} Y} = M_Y(\frac{2t}{\theta})$

$= (1-2t)^{-1}$

$Z \sim Gamma(1, 2) \sim \chi^2_2$

For lower CI.
$Z \geq \chi^2_{2;\alpha}$.

$P(\chi^2_{2;\frac{\alpha}{2}} \leq Z \leq \chi^2_{2;1-\frac{\alpha}{2}}) = 1-\alpha$. $\alpha = 0.1$

$\left(\frac{2Y}{\chi^2_{2;1-\frac{\alpha}{2}}}, \frac{2Y}{\chi^2_{2;\frac{\alpha}{2}}}\right)$