Review of Chap. 7 & 8.

Population distribution $X \sim F_X$. <u>unknown</u>

quantities of interests $\mu = \mathbb{E}X$, $\sigma^2 = \mathrm{Var}\, X$

Samples $X_1, \dots, X_n \overset{i.i.d.}{\sim} F_X$.

<u>Key point</u> Estimation.

$$\hat{\mu} = \bar{X} = \frac{\sum X_i}{n}, \quad \hat{\sigma}^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2.$$

Try to figure out the distribution of the estimator, which is the core of estimation theory.

① F_X is normal $N(\mu, \sigma^2)$.

Normal distribution has nice property.

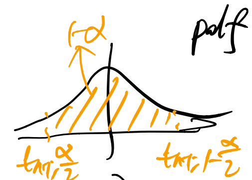
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right). \quad \left\{\begin{array}{l} \mathbb{E}\bar{\mu},\ \mathrm{Var}\,\bar{\mu} \\ P(|\bar{\mu} - \mu| < a) = p. \\ CI \end{array}\right.$$

$$\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}, \quad \bar{X} \perp\!\!\!\perp \hat{\sigma}^2.$$

$$\frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}} = \frac{(\bar{X} - \mu)/\sigma/\sqrt{n}}{\hat{\sigma}/\sigma} \sim t_{n-1}$$

Sometimes, we want to know more about μ other than a single estimator $\hat{\mu}$, like how good it is, what's an interval covers the true value with certain prob. $1-\alpha$.

pdf of Y



Pivotal method for CI.

$$\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \sim t_{n-1} \overset{def}{=} Y \qquad P(Y \in [l, u]) = 1-\alpha.$$

$(-\infty, t_{n-1;1-\alpha}]$

$[t_{n-1;\alpha}, +\infty)$

$[t_{n-1;\frac{\alpha}{2}}, t_{n-1,1-\frac{\alpha}{2}}]$.

$$P\left(\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \in [l, u]\right) = 1-\alpha$$

$$\mu \in [l(\bar{X},\hat{\sigma}), u(\bar{X},\hat{\sigma})]$$

② F_X is general distribution, with finite mean μ and variance σ^2.

Central limit theorem.

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1),$$

but $\mathbb{E}\bar{X}$ and $\mathrm{Var}\,\bar{X}$ can be calculated exactly.

$$\hat{\sigma}^2 \xrightarrow{a.s.} \sigma^2.$$

However, $\bar{X} \not\perp\!\!\!\perp \hat{\sigma}^2$ in general.

Now, everything is "approximate".

We simply regard $\hat{\sigma}^2 = \sigma^2$ when n is large.

So $\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \approx N(0,1)$ $\Big\langle$ $P(|\bar{X}-\mu|<a)=p$; CI.

$$P(|\bar{X}-\mu|<a) = P\Big(\underbrace{\left|\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}}\right|}_{|Z|} < \frac{a}{\hat{\sigma}/\sqrt{n}}\Big) = 1-\underset{\Delta}{\Phi}\Big(-\frac{a}{\hat{\sigma}/\sqrt{n}}\Big).$$

$$P(Z \in [l,u]) = 1-\alpha \qquad l = z_{\frac{\alpha}{2}},\ u = z_{1-\frac{\alpha}{2}}.$$

$$\Leftrightarrow P\Big(\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \in [l,u]\Big) = 1-\alpha$$

↑ "large-sample CI"

↑ approximate CI.

For hw and exam, please

1. Order your answers. It would be great to highlight final results.
2. Even if you can't solve a problem, write the problem number.
3. A result with no steps doesn't deserve pts. It's okay to be simple if all steps are correct, but I encourage moderate explanations on key steps to earn partial pts in case there is some mistake.
4. Good Luck.